Chapter 3

Starting with examples

I am now going to introduce several examples of Riemannian manifolds and attempt to give (or at least discuss) most of the details of a formal definition. I will assume here that the the real numbers $\mathbb{R}$ and the Euclidean spaces $\mathbb{R}^n$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$ consisting of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with each entry x_j a real number for $j = 1, 2, \ldots, n$ are familiar. If you are perceptive enough to know these are actually very mysterious spaces in some ways, do not worry about that.

3.1 Preliminary calculations

My first example(s) involve the case $n = 2$ and the particular set

$$B_1(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

which is the **unit ball** in $\mathbb{R}^2$. So first of all, this set has its usual identity[1] as a (coordinatized Euclidean) subset of $\mathbb{R}^2$ and we're assuming here you know everything (or at least many things) about that set. We will feel free to use any aspects of the Euclidean structure on this set $B_1(\mathbf{0})$ up to and including the calculus of functions with domain the unit ball, though this topic in particular is addressed in some detail elsewhere in these notes. I will simply pause to say that this Euclidean structure we are essentially taking for granted is very crucial to the concept of a Riemannian manifold.

[1]I'm using the term "identity" here in an informal sense, like Superman has his identity as Clark Kent, rather than in any technical sense from algebra.

The idea of our first real example of a Riemannian manifold is that it is, as a set, the same as $B_1(\mathbf{0})$. I will, however, call that set by a different name, and my suggestion is that the new set or structure I introduce bears an identity quite distinct from that of the Euclidean ball. The new name is $\mathcal{B}$. Perhaps I could pick a more distinctive name like M or M^2, but I think $\mathcal{B}$ will do.

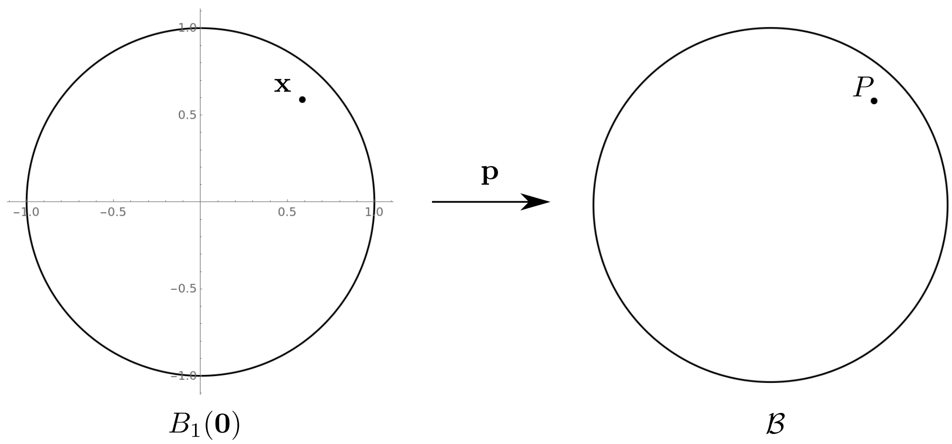

Figure 3.1: The Euclidean unit disk (left) and a Riemann surface $\mathcal{B}$ (right)

In what follows, making the distinction between $\mathcal{B}$ and the Euclidean ball $B_1(\mathbf{0})$ is both crucial and difficult. Before I attempt that crucial and difficult task, I am simply going to suggest some calculations without any notational or conceptual distinction. Then I will try to tease out the identity of $\mathcal{B}$ in stages from there. Each of the calculations involves the assignment of a certain 2×2 matrix to each $\mathbf{x} \in B_1(\mathbf{0})$, namely

$$\begin{pmatrix} \frac{16}{(4+|\mathbf{x}|^2)^2} & 0 \\ 0 & \frac{16}{(4+|\mathbf{x}|^2)^2} \end{pmatrix}. \tag{3.1}$$

I will first describe the suggested calculations in general terms and then give some exercises suggesting specific instances. For the general description, let $a, b \in \mathbb{R}$ be given with $a < b$. Given a path[2] $\alpha \in C^1([a,b] \to B_1(\mathbf{0}))$, recall

[2]Paths are discussed in more detail elsewhere in these notes, but we are assuming some familiarity here. In particular, a path $\alpha \in C^1([a,b] \to B_1(\mathbf{0}))$ as considered here has associated with it two real valued functions $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in C^1[a,b]$, $j = 1, 2$

that the (Euclidean) length is given by

$$\begin{aligned}\text{length}[\alpha] &= \int_a^b |\alpha'(t)|\,dt \\ &= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t)\rangle_{\mathbb{R}^2}}\,dt \\ &= \int_a^b \sqrt{[\alpha_1'(t)]^2 + [\alpha_2'(t)]^2}\,dt.\end{aligned}$$

Both here and in (3.1) the Euclidean norm and innner product (or dot product) are used so that

$$|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x}\rangle_{\mathbb{R}^2} = x_1^2 + x_2^2.$$

For the same path one can calculate the **Riemannian length**. As mentioned above the calculation involves the matrix assignment given in (3.1). For convenience, denote the entries in the matrix by $g_{ij} = g_{ij}(\mathbf{x})$ for $i, j = 1, 2$ so we can also write the matrix as

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} = \frac{16}{(4 + |\mathbf{x}|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Then the Riemannian length is given by

$$\text{length}_{\mathcal{B}}[\alpha] = \int_a^b \sqrt{\langle\ (g_{ij}(\alpha(t)))\ \alpha'(t),\ \ \alpha'(t)\ \rangle_{\mathbb{R}^2}}\,dt.$$

Here the usual conventions for matrix multiplication and the Euclidean inner product in $\mathbb{R}^2$ are used so that

$$\begin{aligned}(g_{ij}(\alpha(t)))\ \alpha'(t) &= \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} \\ &= \begin{pmatrix} g_{11}\alpha_1' + g_{21}\alpha_2' \\ g_{12}\alpha_1' + g_{22}\alpha_2' \end{pmatrix}\end{aligned}$$

and

$$\langle \mathbf{y}, \mathbf{x}\rangle_{\mathbb{R}^2} = y_1x_1 + y_2x_2.$$

Exercise 3.1.1 Find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, a] \to B_1(\mathbf{0})$ by $\alpha(t) = t(\cos\theta, \sin\theta)$ where $\theta \in \mathbb{R}$ and $a > 0$.

Exercise 3.1.2 Find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, \theta] \to B_1(\mathbf{0})$ by $\alpha(t) = a(\cos t, \sin t)$ where $\theta \in \mathbb{R}$ and $a > 0$.

Exercise 3.1.3 For each x, a_0 and θ_0 with $x < 0 < a_0 < 1$ and $0 < \theta_0 \leq \pi/2$ find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, \theta] \to B_1(\mathbf{0})$ by $\alpha(t) = (x, 0) + a(\cos t, \sin t)$ where

$$a = \sqrt{(a_0 \cos \theta_0 - x)^2 + a_0^2 \sin^2 \theta_0}$$

and

$$\theta = \tan^{-1}\left(\frac{a_0 \sin \theta_0}{a_0 \cos \theta_0 - x}\right)$$

as illustrated in Figure 3.2.

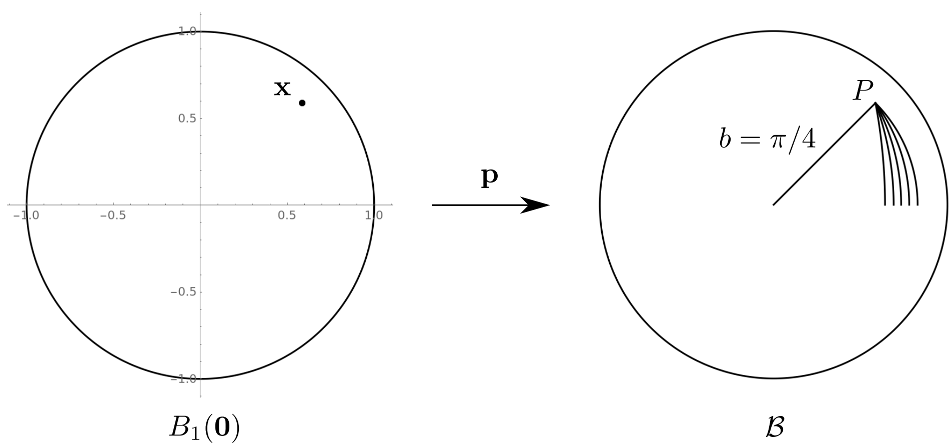

Figure 3.2: The Euclidean unit disk (left) and a Riemann surface $\mathcal{B}$ (right)

Exercise 3.1.4 It is not clear that the notion of the **radius** of a Riemannian manifold makes sense in general, but I'm pretty sure it makes sense to talk about the radius of the Riemannian manifold $\mathcal{B}$. What is the radius of $\mathcal{B}$?

For the second calculation, one is given two paths

$$\alpha \in C^1([a_1, b_1] \to B_1(\mathbf{0})) \qquad \text{and} \qquad \beta \in C^1([a_2, b_2] \to B_1(\mathbf{0}))$$

satisfying $\alpha(t_1) = \beta(t_2)$ for some t_j with $a_j < t_j < b_j$ for $j = 1, 2$. In this case, the paths are said to meet at the (Euclidean) angle $\theta \in [0, \pi]$ if

$$\cos\theta = \frac{\langle \alpha'(t_1), \beta'(t_2) \rangle_{\mathbb{R}^2}}{|\alpha'(t_1)|\ |\beta'(t_2)|}. \tag{3.3}$$

Of course, this does not quite always serve as a definition for the angle at which the paths meet because the value on the right in (3.3) may not be a well-defined real number. Specifically, if $\alpha'(t_1) = \mathbf{0}$ or $\beta'(t_2) = \mathbf{0}$, then there is a problem. If we rule out these possibilities by requiring $|\alpha'(t_1)|\ |\beta'(t_2)| > 0$, then there is no problem.

Exercise 3.1.5 Let $\cos^{-1} : [-1, 1] \to [0, \pi]$ denote the principal arccosine function. Plot $\cos^{-1}$ and explain why this function may be applied to both sides of (3.3).

Under the assumptions described above under which the Euclidean angle between paths α and β is well-defined, the **Riemannian angle** at which the paths α and β meet is given by

$$\theta_{\mathcal{B}} = \cos^{-1}\left(\frac{\langle (g_{ij})\ \alpha'(t_1),\ \beta'(t_2) \rangle_{\mathbb{R}^2}}{\sqrt{\langle (g_{ij})\ \alpha'(t_1),\ \alpha'(t_1) \rangle_{\mathbb{R}^2} \quad \langle (g_{ij})\ \beta'(t_2),\ \beta'(t_2) \rangle_{\mathbb{R}^2}}} \right). \tag{3.4}$$

Exercise 3.1.6 Assuming two paths

$$\alpha \in C^1([a_1, b_1] \to B_1(\mathbf{0})) \qquad \text{and} \qquad \beta \in C^1([a_2, b_2] \to B_1(\mathbf{0}))$$

meet at the Euclidean angle θ as described above, calculate the Riemannian angle $\theta_{\mathcal{B}}$ at which the two paths meet.

The exercise above should have been pretty anticlimactic. Comparison of (3.4) and (3.3) in general might suggest some interesting questions.

Exercise 3.1.7 A **general inner product** on a real vector space V is a function $\langle \cdot\ ,\ \cdot \rangle_V : V \times V \ \to \mathbb{R}$ satisfying the following conditions

(i) $\langle v, w\rangle_V = \langle v, w\rangle_V$ for $v, w \in V$.

(ii) $\langle v, v\rangle_V \geq 0$ for all $v \in V$ with equality if and only if $v = \mathbf{0} \in V$.

(iii) $\langle av + bw, z\rangle_V = a\langle v, z\rangle_V + b\langle w, z\rangle_V$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

If a function $\langle \cdot\ ,\ \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \langle\ (g_{ij})\ \mathbf{v},\ \ \mathbf{w}\ \rangle_{\mathbb{R}^2},$$

where (g_{ij}) is a 2×2 matrix with real entries, then show $\langle \cdot\ ,\ \cdot \rangle$ defines an abstract inner product on $\mathbb{R}^2$ if and only if the following conditions are satisfied by the matrix (g_{ij}):

(i) $g_{ij} = g_{ji}$ for $i, j = 1, 2$,

(ii) $g_{11}, g_{22} > 0$, and

(iii) $g_{11}g_{22} - g_{12}^2 > 0$.

Exercise 3.1.8 A **general bilinear form** on a real vector space V is a function $B : V \times V \to \mathbb{R}$ satisfying the following

(i) $B(av + bw, z) = aB(v, z) + bB(w, z)$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

(ii) $B(z, av + bw) = aB(z, v) + bB(z, w)$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

(a) Show every general bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ determines a unique matrix $(a_{ij}) \in \mathbb{R}^{n\times n}$ for which

$$B(\mathbf{v}, \mathbf{w}) = \langle (a_{ij})\ \mathbf{v},\ \ \mathbf{w}\rangle_{\mathbb{R}^n} \qquad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n. \tag{3.5}$$

(b) Show conversely that given any real matrix $(a_{ij}) \in \mathbb{R}^{n\times n}$ the formula (3.5) determines a unique general bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

The matrix (a_{ij}) is called the **matrix of the bilinear form**.

A general bilinear form $B : V \times V \to \mathbb{R}$ is said to be **symmetric** if $B(v, w) = B(w, v)$ for all $v, w \in V$.

Exercise 3.1.9 A symmetric bilinear form $B : V \times V \to \mathbb{R}$ for which $B(v, w) = 0$ for all $w \in V$ implies $v = \mathbf{0}$ is said to be **nondegenerate**. Show the matrix of a nondegenerate (symmetric) bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is invertible.

The third calculation is most easily described in terms of a **cyclic path**. To get started we can assume a cyclic path is parameterized for some $\epsilon > 0$ by a function $\alpha \in C^1([a-\epsilon, b+\epsilon] \to B_1(\mathbf{0}))$ satisfying the following:

$$\alpha(t_1) \neq \alpha(t_2) \qquad \text{for} \qquad a \leq t_1 < t_2 < b,$$

$$\alpha(t) = \alpha(b-a+t) \qquad \text{for} \qquad a-\epsilon \leq t \leq a,$$
$$\alpha(t) = \alpha(a+t-b) \qquad \text{for} \qquad b \leq t \leq b+\epsilon,$$

and $|\alpha'(t)| \neq 0$. In such a case, the path itself is

$$\Gamma = \{\alpha(t) \in B_1(\mathbf{0}) : t \in [a,b]\}.$$

If $A \subset B_1(\mathbf{0})$ is a region with $\partial A = \Gamma$, then the (Euclidean) area of A is defined to be

$$\text{area}(A) = \int_A 1.$$

The meaning of this integral is assumed to be familiar here but is also discussed in some detail elsewhere in these notes. The **Riemannian area** of the region A enclosed by Γ is given by

$$\text{area}_{\mathcal{B}}(A) = \int_A \sqrt{g_{11}g_{22} - g_{12}^2}.$$

Exercise 3.1.10 For each $a > 0$ calculate the Euclidean area area(A) and the Riemannian area area$_{\mathcal{B}}(A)$ of the region A enclosed by $\alpha : [0, 2\pi] \to B_1(\mathbf{0})$ by $\alpha(t) = a(\cos t, \sin t)$.

Exercise 3.1.11 Notice I only gave a definition for Riemannian area for regions enclosed by C^1 (smooth) paths. Of course, it is not too much to ask to consider the Euclidean and Riemannian areas of the following regions:

(a) $R = \{\mathbf{x} = (x_1, x_2) : a_1 < x_1 < a_2, b_1 < x_2 < b_2\} \subset B_1(\mathbf{0})$.

(b) $S = \{\mathbf{x} = (x_1, x_2) : 0 < x_1 < a, x_2 < mx_1, x_1^2 + x_2^2 < r^2\} \subset B_1(\mathbf{0})$.

(c) $T = \{\mathbf{x} = (x_1, x_2) : \langle \mathbf{x} - \mathbf{x}_j \cdot \mathbf{n}_j \rangle_{\mathbb{R}^2} < 0, j = 1, 2, 3\} \subset \mathcal{B}$.

In these instances a_j and b_j, are appropriate real numbers for $j = 1, 2$, the numbers a, r and m are positive real numbers, and $\mathbf{x}_j$ and $\mathbf{n}_j$ are appropriate elements of $\mathbb{R}^2$ for $j = 1, 2, 3$. Give an appropriate definition of **piecewise** C^1 **cyclic paths** in $B_1(\mathbf{0})$ which allows the regions above to be considered as regions bounded by cyclic paths and regions of integration in particular.

Exercise 3.1.12 Find formulas for the Euclidean areas and Riemannian areas of some of the regions mentioned in **(a)**, **(b)**, and/or **(c)** of Exercise 3.1.11 above.

Exercise 3.1.13 If the Riemannian length of the radial segment indicated in Figure 3.2 is $b = \pi/4$, what is the Euclidean length a of the corresponding segment $\{t(\cos\theta, \sin\theta) \in B_1(\mathbf{0}) : 0 < t < a\}$?

Exercise 3.1.14 For some fixed a_0 and θ_0 in Problem 3.1.3, plot the value of $\text{length}_B[\alpha]$ as a function of x. Do you notice anything interesting?